

# HOLOMORPHIC CORRESPONDENCES BETWEEN CR MANIFOLDS.

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## 1. INTRODUCTION.

One of the interesting phenomena in several complex variables is the analytic continuation of a germ of a biholomorphic map  $f : M \rightarrow M'$ , defined at a point  $p \in M$ , where  $M, M' \subset \mathbb{C}^n$  are real-analytic hypersurfaces. Already Poincaré [Po] observed that a biholomorphic map sending an open piece of a unit sphere in  $\mathbb{C}^2$  to another such open piece must be an automorphism of the unit ball. This was proved for  $\mathbb{C}^n$  in [Ta] and [Al]. Clearly such an extension is possible only for  $n > 1$ , thus showing the very special nature of CR maps between real-analytic CR manifolds.

Pinchuk [Pi1],[Pi3] proved that the germ of a non-degenerate holomorphic map from a real-analytic strictly pseudoconvex hypersurface  $M$  in  $\mathbb{C}^n$  to a unit sphere in  $\mathbb{C}^N$ ,  $N \geq n$ , extends as a holomorphic map along any path on  $M$ . Webster [We] proved that the germ of a biholomorphic map between real-algebraic Levi non-degenerate hypersurfaces in  $\mathbb{C}^n$  extends as an algebraic map, and also gave sufficient conditions for the map to be rational. In that paper he studied the geometric properties of Segre varieties, which were originally introduced in [Se].

Much attention was devoted to the generalization of Webster's theorem to the case of different dimensions and higher codimensions. In this situation both  $M$  and  $M'$  are assumed to be real algebraic submanifolds or sets; that is, defined by the zero locus of a system of real polynomials. Under certain conditions, it then turns out that a locally defined holomorphic map between such objects must necessarily be algebraic. See Baouendi, Ebenfelt and Rothschild [BER1], Huang[Hu], Sharipov and Sukhov [SS], Baouendi, Huang and Rothschild [BHR], Coupet, Meylan and Sukhov [CMS], Zaitsev [Za], Merker [Me1] and many additional references contained therein. The special case of hyperquadrics was also considered in Tumanov [Tu], Forstnerič [Fo], Sukhov [Su] and other papers.

On the other hand, much less is known if at least one of the submanifolds is not assumed to be real-algebraic. In this case the map need not be algebraic; however, analytic continuation along a real-analytic hypersurface is also possible. Pinchuk [Pi2] proved that a germ of a biholomorphic map  $f$  from a strictly pseudoconvex, real-analytic, non-spherical hypersurface  $M$  to a compact, strictly pseudoconvex, real-analytic hypersurface  $M' \subset \mathbb{C}^n$  extends holomorphically along any path on  $M$ . A similar result was shown in [Sh1] for the case when  $M$  is essentially finite, smooth, real-analytic and  $M' \subset \mathbb{C}^n$  is compact, real-algebraic and strictly pseudoconvex. Levi non-degeneracy of the target hypersurface ensures that the extended map is single valued. If the target hypersurface is just assumed to be compact and smooth real-algebraic, the extension in general will be multiple-valued as was proved in [Sh2]. This naturally leads to consideration of holomorphic correspondences, a multiple-valued generalization of holomorphic mappings.

In this paper we study the analytic continuation of germs of holomorphic mappings from smooth real-analytic CR submanifolds of arbitrary codimension to compact smooth real-algebraic generic submanifolds in  $\mathbb{P}^N$  of general codimension. The continuation that we obtain is a holomorphic correspondence from a neighborhood of the submanifold in the pre-image to  $\mathbb{P}^N$ . This is analogous to the algebraicity of the map asserted in the case when both submanifolds are real-algebraic. We also study some applications to maps between pseudoconcave CR submanifolds in  $\mathbb{P}^n$ . It is rather

surprising that under certain conditions, a local CR map between such objects turns out to be the restriction of a rational, or even a linear map in  $\mathbb{P}^n$  without the assumption of algebraicity of the submanifold in the pre-image.

Our results generalize the extension property of a germ of a biholomorphic mapping from a compact real-analytic hypersurface in  $\mathbb{C}^n$  to a compact real-algebraic hypersurface in  $\mathbb{C}^n$  proved in [Sh2]. We remark that the proofs of the main results of this paper differ significantly from those utilized in [Sh2], where the main construction essentially uses the fact that the Segre varieties have codimension one.

In the next section we present the main results. In Section 3 we give some background on CR manifolds, Segre varieties and holomorphic correspondences. Section 4 contains the proof of the local extension of a holomorphic map as a holomorphic correspondence. In Section 5 we prove the global extension. The last section contains applications of the main theorem to pseudoconcave CR submanifolds in  $\mathbb{P}^n$ .

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## 2. STATEMENT OF RESULTS.

**2.1. Analytic Continuation.** By a holomorphic correspondence we mean a complex analytic subset  $A \subset X \times X'$ , where  $X$  and  $X'$  are complex manifolds, such that (i)  $\dim A \equiv \dim X$ , and (ii) the projection  $\pi : A \rightarrow X$  is proper. It is natural to define a multiple valued map  $F = \pi' \circ \pi^{-1} : X \rightarrow X'$  associated with  $A$ . A holomorphic correspondence  $F$  is called a *finite* correspondence, if  $F(p)$  and  $F^{-1}(p')$  are finite sets for any  $p \in X$  and  $p' \in X'$ . We say that  $F$  *splits* at a point  $q \in X$  if there exist a neighborhood  $U_q$  of  $q$  and an integer  $k$  such that  $F|_{U_q}$  is just the union of  $k$  holomorphic maps  $F^j : U_q \rightarrow X'$ ,  $j = 1, \dots, k$ .

Recall that if  $M$  is an abstract paracompact real analytic CR manifold of type  $(m, d)$ , then by [AF] there exists a complex manifold  $X$  of dimension  $n = m + d$  such that  $M$  can be generically embedded into  $X$  as a CR submanifold.

Consider the following situation:  $M$  is a smooth real-analytic minimal CR manifold of type  $(m, d)$ ,  $m, d > 0$ ;  $M' \subset \mathbb{P}^N$  is a compact smooth real-algebraic essentially finite generic submanifold of type  $(m, d')$ ,  $d' > 0$ . The main results of the paper are the following.

**Theorem 2.1.** *Suppose that  $\omega \subset M$  is a relatively compact connected open subset, and  $f : \omega \rightarrow M'$  is a real-analytic CR map such that  $df|_{H_p} : H_p M \rightarrow H_{f(p)} M'$  is an isomorphism between the holomorphic tangent spaces for almost all  $p \in \omega$ . Let  $q \in \partial\omega$ . Then there exists a neighborhood  $U_q \subset X$  of  $q$  such that  $f$  extends to  $U_q$  as a holomorphic correspondence  $F : U_q \rightarrow \mathbb{P}^N$  with  $F(U_q \cap M) \subset M'$ .*

We remark that in Theorem 2.1 we may assume that  $f : \omega \rightarrow M'$  is a smooth CR mapping and the characteristic variety  $\nu_z(f)$  has dimension zero for  $z \in \omega$ , since by the result in [Da],  $f$  is in fact real-analytic. Also note that we do not claim that the extended correspondence near  $q$  is finite.

The Segre map associated with a real-analytic CR manifold  $M$  is defined by  $\lambda : w \rightarrow Q_w$ , where  $Q_w$  is the Segre variety of a point  $w$ . We say that  $\lambda$  is locally injective at a point  $q \in M$ , if there exists a small neighborhood  $U_q$  of  $q$  such that  $\lambda$  is an injective map from  $U_q$  onto its image. For details see Section 3.

**Theorem 2.2.** *Assume in addition that  $M$  is essentially finite and  $d' \geq d$ . Let  $p \in M$ ,  $U_p \subset X$  be a neighborhood of  $p$ , and let  $f : U_p \rightarrow \mathbb{P}^N$  be a holomorphic mapping of maximal rank such*

that  $f(U_p \cap M) \subset M'$ . Suppose that  $M_1$  is a relatively compact simply-connected open subset of  $M$  containing  $p$ . Then

- (1) There exists a neighborhood  $U \subset X$  of  $M_1$  such that  $f$  extends as a finite holomorphic correspondence  $F : U \rightarrow \mathbb{P}^N$  with  $F(M_1) \subset M'$ .
- (2)  $F$  splits into holomorphic mappings at every point  $q \in M_1 \setminus F^{-1}(\Sigma')$ , where  $\Sigma'$  is the set of points on  $M'$  near which the Segre map is not locally injective.

We note that the assumption in Theorem 2.2 that the map  $f$  is of maximal rank ensures that the images of Segre varieties  $Q_z$  under  $f$  and subsequent analytic continuations of  $f$  have the same dimension as  $Q'_z$ . The essential finiteness of  $M'$  then guaranties that the constructed extensions of the map  $f$  are finitely-valued.

To illustrate the conclusions of Theorem 2.2 we consider a simple example. Let  $r$  and  $s$  be coprime positive integers, and  $M$  and  $M'$  be the CR hypersurfaces in  $\mathbb{P}^n$  given in homogeneous coordinates by

$$(1) \quad M = \{z \in \mathbb{P}^n : |z_0|^{2r} + \cdots + |z_k|^{2r} - |z_{k+1}|^{2r} - \cdots - |z_n|^{2r} = 0\},$$

$$(2) \quad M' = \{z' \in \mathbb{P}^n : |z'_0|^{2s} + \cdots + |z'_k|^{2s} - |z'_{k+1}|^{2s} - \cdots - |z'_n|^{2s} = 0\},$$

with  $k < n$ . The finite holomorphic correspondence  $F(z) = (z_0^{r/s}, \dots, z_n^{r/s})$  maps  $M$  to  $M'$ , is  $r^n : s^n$  valued, and  $F$  splits into  $s^n$  holomorphic mappings outside the branching locus of  $F$ . If we choose a point  $p \in M$  outside the branching locus, and if at  $p$  we choose the germ of one of the  $s^n$  branches of  $F$ , then Theorem 2.2 reproduces the entire correspondence  $F$ .

Theorems 2.1 and 2.2 can be considered as a generalization of the results on algebraicity of a local CR map between real-algebraic submanifolds. Non-algebraic holomorphic correspondences can be easily constructed from the examples considered in [BS].

Our proof of Theorems 2.1 and 2.2 is based on the technique of Segre varieties. As it was mentioned in the introduction, Webster was the first to use Segre varieties in the context of holomorphic mappings. His ideas were further developed in [DW], [DF2], [DFY], [DP1] and other papers. Our main construction is based on the technique of these papers. A somewhat different approach was developed in [BJT], [BR1] and [BER1]. We rely on the characterization of minimality in terms of Segre sets proved in [BER1].

Let us briefly explain the idea of the main construction. Let  $p = 0 \in M$ , and let  $U_0$  be a small neighborhood of the origin, where the map  $f$  is defined. The first step is to show that  $f$  extends as a holomorphic correspondence  $F_1$  to a neighborhood  $U_1$  of  $Q_0$ , the Segre variety of the origin. This can be understood as follows. If  $w \in Q_0$ , then  $0 \in Q_w$ , and  $f(U_0 \cap Q_w)$  is a complex subvariety in the target space which passes through  $f(0)$ . If  $f(U_0 \cap Q_w) \subset Q'_{w'}$  for some  $w'$ , then we set  $F_1(w) = w'$ . For  $w$  close to the origin,  $f(Q_w \cap U_0) \subset Q'_{f(w)}$  by the invariance property of Segre varieties, and thus the extension defined in this way agrees with  $f$  on  $U_0$ . For essentially finite manifolds, if  $w'$  is sufficiently close to  $M'$ , then there exists a finite number of points which have the same Segre variety as  $w'$ . Therefore in general  $F_1$  may not be single valued. The graph of  $F_1$  can be described as

$$(3) \quad A_1 = \{(w, w') \in U_1 \times \mathbb{P}^N : f(Q_w \cap U_0) \subset Q'_{w'}\}.$$

The next step of the proof is to define inductively the analytic continuation of  $f$  to bigger sets. This can be achieved using Segre sets (see the next section for definitions). Briefly, once the analytic continuation of  $f$  as a correspondence  $F_1$  is established, we can use a similar construction to define the extension of  $F_1$  along the Segre varieties of points on  $Q_0$ . Let

$$(4) \quad A_2 = \{(w, w') \in U_2 \times \mathbb{P}^N : F_1(Q_w \cap U_1) \subset Q'_{w'}\},$$

where  $U_2$  is a neighborhood of  $Q_0^2 = \cup_{w \in Q_0} Q_w$ . We note that for  $w'$  close to  $M'$  the condition for  $(w, w')$  in (4) roughly speaking means that all branches of  $F_1$  map  $Q_w$  into  $Q_{w'}$ . We can repeat this process for  $A_3$ , using the extension  $A_2$ , etc.

Finally, the crucial observation is that after a certain number of iterations, we obtain an extension of  $f$  to a neighborhood of the origin, which is independent of the choice of  $U_0$ , where  $f$  was originally defined. This follows from the minimality of  $M$  and the property of Segre sets proved in [BER1]. Furthermore, for compact minimal manifolds such neighborhoods can be chosen of uniform size for all points on  $M$ , which allows us to extend  $f$  to any simply-connected relatively compact subset of  $M$ .

The question remains open whether a similar continuation of  $f$  is possible in the case when  $M'$  is real-analytic. This is unknown even in the case when  $M$  is a pseudoconvex hypersurface and  $M'$  is a strictly pseudoconvex hypersurface in  $\mathbb{C}^n$ . Our method heavily relies on the fact that the Segre varieties associated with  $M'$  are globally defined in  $\mathbb{P}^n$ , and therefore cannot be directly extended to the real-analytic case.

**2.2. Pseudoconcave CR manifolds.** Our main application of Theorem 2.2 concerns pseudoconcave CR submanifolds embedded into projective spaces. We recall that a CR manifold is called pseudoconcave, if at each point its Levi form has at least one positive and at least one negative eigenvalue in every characteristic conormal direction.

**Theorem 2.3.** *Consider a connected  $C^\infty$ -smooth compact pseudoconcave CR submanifold  $M$  of  $\mathbb{P}^n$ , having type  $(m, d)$  with  $m, d > 0$ .*

- (a) *Let  $f : M \rightarrow \mathbb{P}^N$  be a continuous CR map. Then  $f$  is the restriction to  $M$  of a rational map  $F : \mathbb{P}^n \rightarrow \mathbb{P}^N$ .*
- (b) *Assume  $n = m + d$ , so that  $M$  is generic in  $\mathbb{P}^n$ . Let  $f : M \rightarrow \mathbb{P}^n$  be a CR map which is a local diffeomorphism onto  $f(M)$ . Then  $f$  is the restriction of a linear automorphism of  $\mathbb{P}^n$ .*

For a locally biholomorphic map from a compact smooth pseudoconcave hypersurface in  $\mathbb{P}^n$  to  $\mathbb{P}^n$  part (b) was first shown in [Iv]. Our proof of Theorem 2.3 is based on the result of [HN5], where it is shown that any CR meromorphic function on  $M$  is necessarily rational.

Combining Theorem 2.3 with Theorem 2.2, we obtain the following results.

**Theorem 2.4.** *Let  $M \subset \mathbb{P}^n$  (resp.  $M' \subset \mathbb{P}^N$ ) be a compact smooth real-analytic pseudoconcave essentially finite CR submanifold of type  $(m, d)$  (resp.  $(m, d')$ ),  $m, d, d' > 0$  and  $d' \geq d$ . Let  $M$  be simply connected and let  $M' \subset \mathbb{P}^N$  be generic real-algebraic, and such that the Segre map associated with  $M'$  is locally injective. Let  $p \in M$ ,  $U_p$  be a neighborhood of  $p$  in  $\mathbb{P}^n$ , and let  $f : U_p \cap M \rightarrow M'$  be a germ of a smooth CR map of maximal rank. Then*

- (a)  *$f$  is the restriction of a rational map  $F : \mathbb{P}^n \rightarrow \mathbb{P}^N$ ;*
- (b) *If  $n = N = m + d$  and the Segre map associated with  $M$  is locally injective, then  $f$  is the restriction of a linear automorphism of  $\mathbb{P}^n$ .*

If  $M, M' \subset \mathbb{P}^n$  are both hypersurfaces, then  $f$  may be assumed to be just a local CR homeomorphism, since in this case  $f$  extends smoothly to a neighborhood of  $p$ , and by [HN2], the Jacobian of  $f$  does not vanish at  $p$ .

**Corollary 2.5.** *If a real-analytic submanifold  $M$ , satisfying the conditions of Theorem 2.4, is locally CR equivalent to a real-algebraic CR submanifold  $M'$  satisfying Theorem 2.4, then  $M$  is necessarily real-algebraic.*

**Remark.** The above results concerning pseudoconcave CR manifolds hold also under a weaker assumption on  $M$  and  $M'$ , namely, instead of pseudoconcavity it is enough to assume that  $M$  and  $M'$  satisfy the so-called Property E. For a generic  $M$  this means that for each  $p \in M$  every local CR function defined near  $p$  extends to a holomorphic function in a full neighborhood of  $p$  in the ambient space. For details see [HN5].

Furthermore, in the special case when  $M'$  is a hyperquadric we obtain the following.

**Theorem 2.6.** *Let  $M$  be a simply-connected compact smooth real-analytic Levi non-degenerate CR manifold of type  $(n-1, 1)$ ,  $n > 1$ . Let  $M'$  be the hyperquadric in  $\mathbb{P}^n$  given in homogeneous coordinates by*

$$(5) \quad |z_0|^2 + |z_1|^2 + \cdots + |z_k|^2 - |z_{k+1}|^2 - \cdots - |z_n|^2 = 0.$$

*Suppose that  $\omega$  is a connected open set in  $M$ , and  $f : \omega \rightarrow M'$  is a CR map that is a local homeomorphism. Then  $M$  and  $M'$  are globally CR equivalent; hence  $M$  has a CR embedding as a hypersurface in  $\mathbb{P}^n$ . In the special case where  $0 < k < n-1$ , and  $M$  is a priori a hypersurface in  $\mathbb{P}^n$ , then  $f$  is the restriction to  $M$  of a linear automorphism of  $\mathbb{P}^n$ .*

Note that Theorem 2.6 includes both the case when  $M'$  is a sphere in  $\mathbb{C}^n$ , which was proved before in [Pi1], and the case when  $M'$  is a compact pseudoconcave hyperquadric. For the sphere our method gives an alternative and independent proof of this well known result.

If  $M$  is not assumed to be simply-connected, then  $f$  in general may not extend to a global map from  $M$  to  $M'$  as examples in [BS] show for the case when  $M'$  is a sphere.

### 3. CR MANIFOLDS, SEGRE VARIETIES AND HOLOMORPHIC CORRESPONDENCES.

An abstract smooth CR manifold of type  $(m, d)$  consists of a connected smooth paracompact manifold  $M$  of dimension  $2m + d$ , a smooth subbundle  $HM$  of  $TM$  of rank  $2m$ , which is called the holomorphic tangent space of  $M$ , and a smooth complex structure  $J$  on the fibers of  $HM$ . Let  $T^{0,1}M$  be the complex subbundle of the complexification  $\mathbb{C}HM$  of  $HM$ , which corresponds to the  $-i$  eigenspace of  $J$ :

$$(6) \quad T^{0,1}M = \{Y + iJY \mid Y \in HM\}.$$

We also require that the formal integrability condition

$$(7) \quad [C^\infty(M, T^{0,1}M), C^\infty(M, T^{0,1}M)] \subset C^\infty(M, T^{0,1}M)$$

holds. We call  $m$  the CR dimension of  $M$  and  $d$  the CR codimension.  $M$  is called minimal at  $p \in M$ , if there exists no local CR manifold  $N \subset M$  passing through  $p$  having CR dimension  $m$ , but strictly smaller real dimension.

The characteristic bundle  $H^0M$  is defined to be the annihilator of  $HM$  in  $T^*M$ . Its purpose is to parametrize the Levi form, which for every  $p \in M$  is defined for  $v \in H_p^0M$  and  $Y \in H_pM$  by

$$(8) \quad \mathcal{L}(v; Y) = d\tilde{v}(Y, JY) = \langle v, [J\tilde{Y}, \tilde{Y}] \rangle,$$

where  $\tilde{v} \in C^\infty(M, H^0M)$  and  $\tilde{Y} \in (M, HM)$  are smooth extensions of  $v$  and  $Y$ . For each fixed  $v$  it is a Hermitian quadratic form for the complex structure  $J_p$  on  $H_pM$ . A CR manifold  $M$  is said to be pseudoconcave if the Levi form  $\mathcal{L}(v, \cdot)$  has at least one negative and one positive eigenvalue for every  $p \in M$  and every nonzero  $v \in H_p^0M$ .

Let  $M$  and  $M'$  be two abstract smooth CR manifolds, with holomorphic tangent spaces  $HM$  and  $HM'$ . A smooth map  $f : M \rightarrow M'$  is CR if  $f_*(HM) \subset HM'$ , and  $f_*(Jw) = J'f_*(w)$  for every  $w \in HM$ . A CR embedding of an abstract CR manifold  $M$  into a complex manifold  $X$  is a

CR map which is an embedding. We say that the embedding is *generic* if the complex dimension of  $X$  is  $(m + d)$ .

If  $M$  is a real-analytic CR manifold of type  $(m, d)$ , then by [AH],  $M$  is locally CR embeddable. Furthermore, by [AF] there exists a complex manifold  $X$  such that  $M$  can be globally generically embedded into  $X$ . Consider a connected open set  $\omega$  on  $M$ . When  $M$  is real-analytic, and  $f$  is a real-analytic CR function in  $\omega$ , then there is a connected open set  $\Omega_f$  in  $X$ , with  $\omega = M \cap \Omega_f$ , and a holomorphic extension  $\tilde{f}$  of  $f$  to  $\Omega_f$ . When  $M$  is a generic  $C^\infty$ -smooth pseudoconcave CR submanifold of  $X$ , then there exists a connected open set  $\Omega$  in  $X$ , with  $\omega = M \cap \Omega$ , such that any CR distribution in  $\omega$  has a unique holomorphic extension to  $\Omega$ . See [BP], [NV], [HN3], [HN4].

Most of our considerations of real-analytic CR manifolds will be local, and therefore by the above mentioned results we can assume without loss of generality that  $M$  is a generically embedded CR submanifold of some open set in  $\mathbb{C}^n$ , where  $n = m + d$ . Note here that a compact pseudoconcave CR manifold cannot be embedded (even non-generically) into any Stein manifold (see e.g. [HN1]). Therefore, for application purposes the main theorem is formulated for the target submanifold embedded into  $\mathbb{P}^N$ .

Let  $M$  be a generic smooth real-analytic submanifold of  $\mathbb{C}^n$  and let  $p \in M$ . Then in a sufficiently small neighborhood  $U$  of  $p$ ,  $M$  is given by

$$(9) \quad M = \{z \in U : \rho_j(z, \bar{z}) = 0, \quad j = 1, \dots, d\},$$

where each  $\rho_j$  is a real-valued real-analytic function and

$$(10) \quad \bar{\partial}\rho_1 \wedge \dots \wedge \bar{\partial}\rho_d \neq 0 \text{ on } M \cap U.$$

We set  $\rho = (\rho_1, \rho_2, \dots, \rho_d)$ . There exists a biholomorphic change of coordinates near  $p$ ,  $z = (\xi, \zeta) \in \mathbb{C}^m \times \mathbb{C}^d = \mathbb{C}^n$  such that in the new coordinates,  $p = 0$ , and  $M$  is given by

$$(11) \quad \text{Im } \zeta = \phi(\xi, \bar{\xi}, \text{Re } \zeta),$$

where  $\phi$  is a vector-valued real-analytic function with  $\phi(0) = 0$  and  $d\phi(0) = 0$ .

If  $U$  is sufficiently small, to every point  $w \in U$  we can associate to  $M$  its so-called Segre variety in  $U$  defined as

$$(12) \quad Q_w = \{z \in U : \rho_j(z, \bar{w}) = 0, \quad j = 1, \dots, d\},$$

where  $\rho_j(z, \bar{w})$  is the complexification of the defining functions of  $M$ . Another important variety associated with the submanifold  $M$  and the neighborhood  $U$  is

$$(13) \quad I_w = \{z \in U : Q_w = Q_z\}.$$

From the reality condition on the defining functions the following simple but important properties of Segre varieties follow:

$$(14) \quad z \in Q_w \Leftrightarrow w \in Q_z,$$

$$(15) \quad z \in Q_z \Leftrightarrow z \in M,$$

$$(16) \quad w \in M \Leftrightarrow I_w \subset M.$$

If  $0 \in M$ , then from (10) and the implicit mapping theorem, there exist a local change of coordinates near the origin, and a pair of small neighborhoods  $U$  and  $U_0$  of the origin,  $U \Subset U_0$ , where  $U_0$  is given in the product form

$$(17) \quad U_0 = {}'U_0 \times \tilde{U}_0, \quad {}'U_0 \subset \mathbb{C}^m, \quad \tilde{U}_0 \subset \mathbb{C}^d,$$

such that for every  $w \in U$ , the set  $Q_w \cap U_0$  can be represented as the graph of a holomorphic mapping. That is

$$(18) \quad Q_w \cap U_0 = \{z = (\xi, \zeta) \in {}'U_0 \times \tilde{U}_0 : \zeta = h(\xi, \bar{w})\},$$

where  $h(\xi, \bar{w})$  is holomorphic in  $\xi$  and  $\bar{w}$ . Thus  $Q_w$  is a complex submanifold of  $U$  of complex codimension  $d$ .

The main use of Segre varieties comes from the fact that they are invariant under biholomorphic mappings. More precisely, given a holomorphic map  $f : U \rightarrow U'$ , sending a generic smooth real-analytic submanifold  $M$  to another such submanifold  $M'$ ,  $f(p) = p'$  implies  $f(Q_p \cap U) \subset Q_{p'}$  for  $p$  sufficiently close to  $M$ . An analogous property holds also for holomorphic correspondences.

The proof of the basic properties of Segre varieties in higher codimensions is similar to the hypersurface case and can be found in [BER2] or [Me2].

A real analytic submanifold  $M$  is called essentially finite at  $p \in M$ , if  $I_p = \{p\}$  in a small neighborhood of  $p$ . The *Segre map* is defined by  $l : w \rightarrow Q_w$ . A manifold  $M$  being essentially finite now means that the Segre map is finite near  $M$ . It can be shown (see e.g. [Me2]) that any generic Levi non-degenerate CR submanifold of  $\mathbb{C}^n$  is essentially finite. Moreover, if  $M$  is a compact generic submanifold of  $\mathbb{C}^n$ , then it is automatically essentially finite, since by [DF1], any compact real-analytic subset of  $\mathbb{C}^n$  does not contain any non-trivial germs of complex-analytic varieties.

We say that  $\lambda$  is locally injective at a point  $q \in M$ , if there exists a small neighborhood  $U_q$  of  $q$  such that  $\lambda$  is an injective map from  $U_q$  onto its image. It is easy to see that for any Levi non-degenerate hypersurface in  $\mathbb{C}^n$  the Segre map is locally injective.

In [BER1] the authors introduced so-called Segre sets. We briefly recall this construction here. Let  $M$  be a generic smooth real-analytic submanifold of  $\mathbb{C}^n$ ,  $0 \in M$  and let  $Q_0 = Q_0^1$  be the usual Segre variety of 0 as defined in (12). Define

$$(19) \quad Q_0^j = \bigcup_{z \in Q_0^{j-1}} Q_z, \quad j > 1.$$

Then

$$(20) \quad Q_0^1 \subset Q_0^2 \subset \cdots \subset Q_0^j \subset \cdots.$$

Indeed, let  $k$  be the smallest integer such that  $Q_0^k \not\subset Q_0^{k+1}$ . Clearly,  $k \geq 2$ . If  $z \in Q_0^k \setminus Q_0^{k+1}$ , then there exists  $w \in Q_0^{k-1}$  such that  $z \in Q_w$ . By assumption,  $Q_0^{k-1} \subset Q_0^k$ . Therefore  $w \in Q_0^k$ , and  $Q_w \subset Q_0^{k+1}$ ; in particular  $z \in Q_0^{k+1}$ , which is a contradiction.

According to [BER1] (see also [BER2] and [Me3] for a short proof of this fact), there exists an integer  $j_0$ ,  $0 < j_0 < \infty$ , such that  $\bigcup_{j \leq j_0} Q_0^j$  contains a neighborhood of the origin in  $\mathbb{C}^n$ , provided that  $M$  is minimal at 0. We define

$$(21) \quad \emptyset_0 = \{z : z \in Q_0^j, \quad j \leq j_0\}.$$

Moreover, if  $M$  is compact, or is a relatively compact open set of a bigger minimal submanifold, then there exists  $\epsilon > 0$  such that for any point  $p \in M$ , the neighborhood  $\emptyset_p$ , defined as in (21), contains a ball of radius  $\epsilon$  centered at  $p$ .

Suppose now that the manifold  $M \subset \mathbb{P}^n$  is connected and defined by real polynomials. Then the Segre varieties associated with  $M$  can be defined globally as projective algebraic varieties in  $\mathbb{P}^n$ . Indeed, let  $M \cap \mathbb{C}^n$  be given as a connected component of the set defined by

$$(22) \quad \{z \in \mathbb{C}^n : \rho_j(z, \bar{z}) = 0, \quad j = 1, \dots, d\},$$

where  $\rho_j$  are real polynomials. We can projectivize each  $\rho_j$  to define  $M$  in  $\mathbb{P}^n$  in homogeneous coordinates

$$(23) \quad \hat{z} = [\hat{z}_0 : \hat{z}_1 : \cdots : \hat{z}_n], \quad z_k = \frac{\hat{z}_k}{\hat{z}_0}, \quad k = 1, \dots, n,$$

as a connected component of the set defined by

$$(24) \quad \{\hat{z} \in \mathbb{P}^n : \hat{\rho}_j(\hat{z}, \bar{\hat{z}}) = 0\}.$$

We may define now the *polar* of  $M$  as

$$(25) \quad \hat{M}^c = \{(\hat{z}, \hat{\zeta}) \in \mathbb{P}^n \times \mathbb{P}^n : \hat{\rho}_j(\hat{z}, \hat{\zeta}) = 0, \quad j = 1, \dots, d\}.$$

Then  $\hat{M}^c$  is a complex algebraic variety in  $\mathbb{P}^n \times \mathbb{P}^n$ . Given  $\tau \in \mathbb{P}^n$ , we set

$$(26) \quad \hat{Q}_\tau = \hat{M}^c \cap \{(\hat{z}, \hat{\zeta}) \in \mathbb{P}^n \times \mathbb{P}^n : \hat{\zeta} = \bar{\tau}\}.$$

We define the projection of  $\hat{Q}_\tau$  to the first coordinate to be the Segre variety of  $\tau$ .

Alternatively, given (22) one can define the polar as

$$(27) \quad M^c = \{(z, \zeta) \in \mathbb{C}^n \times \mathbb{C}^n : \rho_j(z, \zeta) = 0, \quad j = 1, \dots, d\}.$$

The submanifold  $M \cap \mathbb{C}^n$  can be recovered by intersecting  $M^c$  with the totally real subspace  $\mathcal{T} = \{(z, \zeta) \in \mathbb{C}^{2n} : \zeta = \bar{z}\}$  and taking an appropriate connected component. Given  $w \in \mathbb{C}^n$ , we define

$$(28) \quad Q_w^c = M^c \cap \{(z, \zeta) \in \mathbb{C}^{2n} : \zeta = \bar{w}\}.$$

The standard Segre variety  $Q_w$  can now be recovered by projecting  $Q_w^c$  to  $\mathbb{C}_z^n$ . The algebraic varieties  $M^c$  and  $Q_w^c$  can be projectivized which gives us objects geometrically equivalent to (25) and (26). Note that the closure  $\bar{\mathcal{T}} \subset \mathbb{P}^{2n}$  of the set  $\mathcal{T}$  is a smooth submanifold of  $\mathbb{P}^{2n}$ , and thus  $M$  can be identified with a connected component of  $\bar{M}^c \cap \bar{\mathcal{T}}$ .

We note that condition (10) implies that for  $w$  close to  $M$ , the Segre variety  $Q_w$  contains a connected component  $\tilde{Q}_w$  of dimension  $m$ . However in general,  $Q_w$  may have other components, which a priori may even have different dimension (higher than  $m$ ). For  $w \in M$ ,  $w \in \tilde{Q}_w$ , and for  $w$  close to  $M$ ,  $\tilde{Q}_w$  is the component which near  $w$  is given by (18).

We will understand essential finiteness of a real algebraic submanifold  $M \subset \mathbb{P}^n$  in the sense that for every  $w \in M$ , the set

$$(29) \quad \tilde{I}_w := \{z \in M : \tilde{Q}_z = \tilde{Q}_w\}$$

is finite. In this case we can further show that generically the various  $\tilde{I}_w$  have the same number of points. Note that the set  $\tilde{I}_w$  is a globally defined object in  $\mathbb{P}^n$ .

**Lemma 3.1.** *Let  $M$  be a compact smooth real-algebraic essentially finite generic CR submanifold in  $\mathbb{P}^n$ . Then there exist a neighborhood  $U \subset \mathbb{P}^n$  of  $M$  and an integer  $R \geq 1$  such that*

$$(30) \quad \#\tilde{I}_w = \#\{z \in U : \tilde{Q}_z = \tilde{Q}_w\} = R$$

for almost all  $w \in U$ .

*Proof.* For each  $p \in M$  choose a product neighborhood  $U_p$  as in (17). Let  $U = \bigcup_{p \in M} U_p$ . We define the map  $l$  from  $U$  to the set of all algebraic varieties in  $\mathbb{P}^n$  of dimension  $m$  by letting  $l(w) := \tilde{Q}_w$ . Then  $\tilde{Q}_w$  depends anti-holomorphically on  $w$ . Locally, near almost every  $w \in U$ , the varieties  $\tilde{Q}_w$  have the same algebraic degree. Since algebraic varieties of positive codimension do not divide  $U$ , the  $\tilde{Q}_w$  have fixed degree for almost all  $w$  in  $U$ . Algebraic varieties of fixed dimension and degree are known to be parametrized by the so-called Chow variety (see e.g. [Ha]), and the parametrization  $l$  is an algebraic map between algebraic varieties.



Let  $Y = l(U)$ . Then since  $M$  is essentially finite,  $\dim Y = n$ . It follows (see e.g. [Mu]) that there exists an algebraic variety  $Z \subset Y$  such that for any  $q \in Y \setminus Z$ ,

$$(31) \quad \#l^{-1}(q) = \deg(l) := R,$$

where  $R$  is a positive integer. From (31) the assertion follows.  $\square$

Let  $X$  and  $X'$  be complex manifolds and  $D \subset X$ ,  $D' \subset X'$  be open sets. Recall that if  $A \subset D \times D'$  is a holomorphic correspondence, then  $\pi : A \rightarrow D$  is proper.  $A$  is called a proper holomorphic correspondence, if  $\pi' : A \rightarrow D'$  is also proper. Throughout the paper we will identify the multiple valued map  $F := \pi' \circ \pi^{-1}$  with its graph  $A$ , so given a point  $p \in D$ ,

$$(32) \quad F(p) = \{p' \in D' : \pi' \circ \pi^{-1}(p)\}$$

is a compact subset of  $D'$ . Given a complex analytic subset  $G' \subset D'$ ,  $F^{-1}(G')$  is a complex analytic subset of  $D$ . Indeed  $\pi'^{-1}(G')$  is clearly analytic, and since  $\pi$  is proper, it follows from Remmert's theorem that  $\pi(\pi'^{-1}(G'))$  is analytic in  $D$ .  $F$  is called a *finite-valued* holomorphic correspondence if  $F(p)$  is a finite set for any  $p \in D$ , and a *finite* correspondence if in addition  $F^{-1}(p')$  is finite for any  $p' \in D'$ . If  $X'$  is a Stein manifold, then any proper holomorphic correspondence  $F$  is automatically finite-valued. To see this observe that by the proper embedding theorem,  $X'$  can be viewed as a submanifold of  $\mathbb{C}^{n'}$  for some  $n' > 1$ , and a compact complex analytic set  $F(p)$  must be discrete. Similarly, if  $X$  and  $X'$  are both Stein, then any proper holomorphic correspondence is finite.

Given a finite-valued holomorphic correspondence  $A \subset D \times D'$ , there exists a complex subvariety  $S \subset D$  (possibly empty) such that for any point  $p \in D \setminus S$ , there exists a neighborhood  $U_p \subset D \setminus S$ , such that  $F$  splits into  $k$  holomorphic maps  $F^j : U_p \rightarrow D'$ ,  $j = 1, \dots, k$ , that represent  $F$ . The integer  $k$  is independent of  $p$ , and the  $F^j$  are called the *branches* of  $F$ .

Given a locally complex analytic set  $A$  in  $X$  of pure dimension  $p$ , we say that  $A$  *extends analytically* to an open set  $U \subset X$ ,  $A \cap U \neq \emptyset$ , if there exists a (closed) complex-analytic set  $A^*$  in  $U$  such that (i)  $\dim A^* \equiv p$ , (ii)  $A \cap U \subset A^*$  and (iii) every irreducible component of  $A^*$  has a nonempty intersection with  $A$  of dimension  $p$ . By the uniqueness theorem for analytic sets such analytic continuation of  $A$  is uniquely defined. From this we define the analytic continuation of holomorphic correspondences:

**Definition 3.2.** Let  $D \subset X$  and  $D' \subset X'$  be open sets and let  $A \subset D \times D'$  be a holomorphic correspondence. We say that  $A$  extends as a holomorphic correspondence to an open set  $U \subset X$ ,  $D \cap U \neq \emptyset$ , if there exists an open set  $U' \subset X'$  such that  $A \cap (U \times U') \neq \emptyset$ ,  $A$  extends analytically to a set  $A^* \subset U \times U'$ , and  $\pi : A^* \rightarrow U$  is proper.

If  $A$  and  $A^*$  are both finite-valued, then  $A^*$  may have more branches in  $D \cap U$  than  $A$ . The following lemma gives a simple criterion for the extension to have the same number of branches.

**Lemma 3.3.** Let  $A^* \subset U \times X'$  be a finite-valued holomorphic correspondence which is an analytic extension of a finite correspondence  $A \subset D \times D'$ . Suppose that for any  $z \in (D \cap U)$ ,

$$(33) \quad \#\{\pi^{-1}(z)\} = \#\{\pi^{*-1}(z)\},$$

where  $\pi : A \rightarrow D$  and  $\pi^* : A^* \rightarrow U$  are the projections. Then  $A \cup A^*$  is a holomorphic correspondence in  $(D \cup U) \times X'$ .

The proof is the same as in [Sh2], Lemma 2.

## 4. LOCAL CONTINUATION AS A CORRESPONDENCE.

In this section we prove Theorem 2.1. We may assume the following situation:  $\emptyset \subset \mathbb{C}^n$  is a connected open set,  $\emptyset \cap M = \omega$ ,  $f : \emptyset \rightarrow \mathbb{P}^N$  is a holomorphic map, and  $f(\emptyset \cap M) \subset M'$ . Let  $q \in \partial\emptyset \cap M$  and let  $U$  be a neighborhood of  $q$  such that for any  $z \in U$ , the Segre variety  $Q_z$  is defined in a strictly larger set, and can be represented as in (18). We show that  $f$  extends to some neighborhood of  $q$  as a holomorphic correspondence. We choose a point  $a \in \omega$  such that  $df|_{H_a M}$  is an isomorphism, and so close to  $q$  that  $q \in \emptyset_a$ , where  $\emptyset_a$  is defined as in (21). Fix some neighborhood  $U_a \subset \Omega$  of  $a$ . We will show that  $f|_{U_a}$  extends as a holomorphic correspondence to  $\emptyset_a$ , in particular to a neighborhood of  $q$ .

Assume for simplicity that  $a = 0$ . Choose a small neighborhood  $U_0$  of the origin and shrink  $U_a$  in such a way, that for any  $w \in U_0$ , the set  $Q_w \cap U_a$  is non-empty and connected. Let  $U' \subset \mathbb{P}^N$  be the neighborhood of  $M'$  as in Lemma 3.1. Define

$$(34) \quad A_0 = \{(w, w') \in U_0 \times U' : f(Q_w \cap U_a) \subset Q'_{w'}\}.$$

Then  $A_0$  is a complex-analytic subset of  $U_0 \times U'$ . In the case when  $M$  and  $M'$  are hypersurfaces, this was shown in [Sh2], the proof in the general case is analogous (see also similar constructions later in this section). Note that in (34),

$$(35) \quad \dim Q_w = \dim Q'_{w'} = \dim f(Q_w \cap U_a) = m.$$

Indeed, since  $df|_{H_0 M}$  is an isomorphism,  $\dim Q_0 = \dim f(Q_0 \cap U_0)$ . Without loss of generality we may assume that  $M$  near 0 and  $M'$  near  $f(0)$  are chosen as in (11). Let  $J_f(z)$  be the Jacobian matrix associated with the map  $f$ . Then in the chosen coordinate systems,  $df|_{H_0 M}$  being an isomorphism means that the principal minor of  $J_f(0)$  of size  $m \times m$  has a non-zero determinant. The same property also holds for points sufficiently close to the origin. Therefore, after shrinking  $U_0$ , (35) holds for all  $w$  in  $U_0$ . After further shrinking  $U_0$  if necessary, we may assume that if  $(w, w') \in A_0$ , then  $w' \in \tilde{I}'_{f(w)}$ , which is a finite set by the assumption on  $M'$ . It follows that  $\dim A_0 \equiv n$ . Finally, (16) implies that if  $U_0$  is sufficiently small, then  $A_0$  has no limit points on  $U_0 \times \partial U'$ , and therefore the natural projection from  $A_0$  to the first coordinate is proper. This shows that  $A_0$  defines a holomorphic correspondence. Denote the corresponding multiple valued map by  $F_0$ .

We shrink the neighborhood  $U_0$  of the origin, where  $F_0$  is defined, and choose a 'thin' neighborhood  $U_1$  of  $Q_0 \cap U$  such that for any  $w \in U_1$ , the set  $Q_w \cap U_0$  is non-empty and connected. Note that from (14),  $0 \in Q_w$  for any  $w \in Q_0$ . Denote now by  $A_0$  the graph of  $F_0$  in  $U_0 \times \mathbb{P}^N$ . We define the set  $A_1$  as follows:

$$(36) \quad A_1 = \{(w, w') \in U_1 \times \mathbb{P}^N : F_0(Q_w \cap U_0) \subset Q'_{w'}\}.$$

Then  $A_1$  is a complex-analytic subset of  $U_1 \times \mathbb{P}^N$ . To verify this assertion we prove the following:

1.  $A_1 \neq \emptyset$ . Indeed, by the invariance property of Segre varieties,  $f(Q_w \cap U_0) \subset Q'_{f(w)}$  for  $w$  sufficiently close to the origin. Then

$$(37) \quad F_0(Q_w \cap U_0) \subset Q'_{w'},$$

where  $w' \in F_0(w)$ . To see this, suppose that  $z \in Q_w \cap U_0$  and  $z' \in F_0(z)$ . By construction,  $F_0(Q_z \cap U_0) \subset Q'_{z'}$ . If  $w$  is sufficiently close to the origin,  $w \in Q_z \cap U_0$ , and therefore  $w' \in F_0(w) \subset Q'_{z'}$ . This implies  $z' \in Q'_{w'}$ . And since  $z \in Q_w \cap U_0$  was arbitrary, (37) holds. Moreover,  $F_0(w) = \tilde{I}'_{w'}$ , and thus

$$(38) \quad A_0|_{(U_1 \cap U_0) \times \mathbb{P}^N} \subset A_1,$$

in particular  $A_1$  is non-empty.

2.  $A_1$  is a complex-analytic set in a neighborhood of any of its points. Indeed, let  $(w_0, w'_0) \in A_1$ ,  $z_0 \in Q_{w_0} \cap U_0$  be an arbitrary point, and let  $U_{z_0} \subset U_0$  be a small neighborhood of  $z_0$ . Since  $Q_w \cap U_0$  is connected for all  $w \in U_1$ ,  $F_0(Q_w \cap U_{z_0}) \subset Q'_{w'}$  implies  $F_0(Q_w \cap U_0) \subset Q'_{w'}$ . Therefore in (36)  $U_0$  can be replaced with  $U_{z_0}$ . Choose  $U_{w_0}$  so small that  $Q_w \cap U_{z_0} \neq \emptyset$  for all  $w \in U_{w_0}$ . Since the pre-image of an analytic set under a holomorphic correspondence is an analytic set,  $S_{w'} := F_0^{-1}(Q'_{w'})$  is an analytic subset of  $U_0$ . Let  $U'_{w'_0}$  be so small that  $S_{w'} \cap U_{z_0} \neq \emptyset$  for all  $w' \in U'_{w'_0}$ . Let  $S_{w'}$  near  $z_0$  be given by

$$(39) \quad S_{w'} = \{z \in U_{z_0} : \phi_j(z, \bar{w}') = 0, j = 1, \dots, \tilde{j}\},$$

where the  $\phi_j$  depend holomorphically on  $\bar{w}'$ . We may assume that  $U_{z_0}$  is chosen as in (17), and therefore  $z \in Q_w$  simply means  $z = (\xi, \zeta)$  and  $\zeta = h(\xi, \bar{w})$ . Then the condition  $F_0(Q_w \cap U_{z_0}) \subset Q'_{w'}$  is equivalent to

$$(40) \quad \phi_j((\xi, h(\xi, \bar{w})), \bar{w}') = 0, \quad \xi \in U_{z_0}, j = 1, \dots, \tilde{j}.$$

This is an infinite system of holomorphic equations (after conjugation) which defines  $A_1$  as an analytic set in  $U_{w_0} \times U'_{w'_0}$ .

3.  $A_1$  is closed in  $U_1 \times \mathbb{P}^N$ . Indeed, suppose that  $(w^j, w'^j) \rightarrow (w^0, w'^0)$ , as  $j \rightarrow \infty$ , where  $(w^j, w'^j) \in A_1$  and  $(w^0, w'^0) \in U_1 \times \mathbb{P}^N$ . Since  $Q_{w^j} \rightarrow Q_{w^0}$ , and  $Q'_{w'^j} \rightarrow Q'_{w'^0}$  as  $j \rightarrow \infty$ , by analyticity  $F_0(Q_{w^0} \cap U_0) \subset Q'_{w'^0}$ , which implies that  $(w^0, w'^0) \in A_1$  and thus  $A_1$  is a closed set.

It follows from 1–3 that  $A_1$  is a complex-analytic subset of  $U_1 \times \mathbb{P}^N$ . Let  $\pi_1 : A_1 \rightarrow U_1$  and  $\pi'_1 : A_1 \rightarrow \mathbb{P}^N$  be the coordinate projections. Since  $\mathbb{P}^N$  is compact,  $\pi_1$  is proper. We consider only the irreducible components of  $A_1$  of dimension  $n$  which contain  $A_0$ . The union of all such components we denote again by  $A_1$ . Thus  $A_1$  is an analytic continuation of  $A_0$  as a holomorphic correspondence. Denoting  $U_0 \cap U_1$  again by  $U_0$ , we may assume from the uniqueness theorem for analytic sets that

$$(41) \quad A_1|_{U_0 \times \mathbb{P}^N} = A_0.$$

We set  $F_1 := \pi'_1 \circ \pi_1^{-1} : U_1 \rightarrow \mathbb{P}^N$ .

By construction, if  $(w, w') \in A_1$ , then for any  $(z, z') \in A_1$  such that  $z \in Q_w \cap U_0$ , we necessarily have  $z' \in Q'_{w'}$ . In order to construct analytic sets  $A_j$  which will extend  $A_1$ , we wish to conclude the same for all  $z \in Q_w \cap U_1$ . The difficulty is that in general,  $Q_w \cap U_1$  may have more than one connected component. To prove the assertion we argue by contradiction, and assume that there exists a point  $(w^0, w'^0) \in A_1$  and  $(z, z') \in A_1$ , such that  $z \in Q_{w^0} \cap U_1$ , but  $z' \notin Q'_{w'^0}$ . Connect  $w^0$  and the origin with a smooth path  $\gamma$  contained in  $U_1$ . We may assume that  $w^0$  is the first point on  $\gamma$  for which the desired property does not hold. Without loss of generality we may also assume that for all  $p \in \gamma$  between the origin and  $w^0$  (excluding  $w^0$ ) there exists a small neighborhood  $U_p$  such that whenever  $z \in U_p$ , all components of  $Q_z \cap U_1$  are mapped by  $F_1$  into the same Segre variety. For each point  $p \in \gamma$  between the origin and  $w^0$ , we construct the following set:

$$(42) \quad A_p = \{(w, w') \in U(Q_p) \times \mathbb{P}^N : F_1(Q_w \cap U_p) \subset Q'_{w'}\},$$

where  $U_p \subset U_1$  is a neighborhood of  $p$  and  $U(Q_p)$  is a neighborhood of  $Q_p \cap U$ , which are chosen in such a way that  $U_p$  satisfies the property described above and that for any  $w \in U(Q_p)$ , the set  $Q_w \cap U_p$  is connected. Repeating the argument that was used for  $A_1$ , one can prove that each  $A_p$

is a complex analytic set, which defines a holomorphic correspondence. For  $p = 0$ , this is just the set  $A_1$ . Moreover, for any  $p$  between the origin and  $w^0$  we have

$$(43) \quad A_1|_{(U(Q_p) \cap U_1) \times \mathbb{P}^N} \subset A_p|_{(U(Q_p) \cap U_1) \times \mathbb{P}^N}.$$

Indeed, suppose that  $w \in U(Q_p) \cap U_1$  and  $(w, w') \in A_1$ . Let  $z \in Q_w \cap U_p$  be an arbitrary point, and  $z' \in F_1(z)$ . Then  $F_0(Q_z \cap U_0) \subset Q'_{z'}$ . From (41) we have  $F_1(Q_z \cap U_0) \subset Q'_{z'}$ . By the assumption on  $U_p$ ,  $F_1(Q_z \cap U_1) \subset Q'_{z'}$ , in particular  $F_1(w) \subset Q'_{z'}$ . Therefore,  $w' \in Q'_{z'}$ , and  $z' \in Q'_{w'}$ . Since  $z \in Q_w \cap U_p$  was arbitrary, it follows that  $F_1(Q_w \cap U_p) \subset Q'_{w'}$ . But this means that  $(w, w') \in A_p$ , and thus (43) holds.

For any  $p$ ,  $Q_p \cap U$  is a connected set in  $U(Q_p)$  and therefore is mapped by  $A_p$  into the same Segre variety. By continuity and from (43) we conclude that  $F_1(Q_{w_0} \cap U_1) \subset Q'_{w'_0}$ , which contradicts the assumption. Thus for any  $(w, w') \in A_1$ , if  $(z, z') \in A_1$  and  $z \in Q_w \cap U_1$ , then  $z' \in Q'_{w'}$ .

We now define recursively for  $j > 1$  the following sets:

$$(44) \quad A_j = \{(w, w') \in U_j \times \mathbb{P}^N : F_{j-1}(Q_w \cap U_{j-1}) \subset Q'_{w'}\}.$$

Here the open set  $U_j$  is defined as follows. Suppose that the set  $A_{j-1} \subset U_{j-1} \times \mathbb{P}^N$  is already defined and  $U_{j-1} \subset U$  is some connected open set. We let  $U_j$  be the set of points  $w$  in  $U$  such that  $Q_w \cap U_{j-1} \neq \emptyset$ . Furthermore, after shrinking at each step, if necessary, the sets  $U_k$  for  $k < j$ , we may assume that  $U_{k-1} \subset U_k$  for  $1 \leq k \leq j$ . Note that it follows from the construction that  $Q_0^k \subset U_k$  for  $1 \leq k \leq j$ .

We claim that for all  $j > 0$ ,  $A_j$  is a complex-analytic subset of  $U_j \times \mathbb{P}^N$ , which satisfies the following properties:

- (i)  $A_j|_{U_{j-1} \times \mathbb{P}^N} = A_{j-1}$ ;
- (ii)  $A_j$  defines a holomorphic correspondence  $F_j : U_j \rightarrow \mathbb{P}^N$ ;
- (iii) for any  $(w, w') \in A_j$ , if  $(z, z') \in A_j$  and  $z \in Q_w \cap U_j$ , then  $z' \in Q'_{w'}$ .

Condition (iii) can be understood in the sense that the map  $F_j$ , associated with  $A_j$ , sends all connected components of  $Q_w \cap U_j$  into  $Q'_{w'}$  provided that  $(w, w') \in A_j$ .

The proof is by induction. The case  $j = 1$  is already proved. Suppose that  $A_{j-1}$  is as claimed. We show that the set defined by (44) is also a holomorphic correspondence satisfying properties (i)–(iii).

(i) Let  $w \in U_{j-1}$ , and  $(w, w') \in A_{j-1}$ . Then by definition,  $F_{j-2}(Q_w \cap U_{j-2}) \subset Q'_{w'}$ . From property (i), which by the induction hypothesis holds for  $F_{j-1}$ , the correspondences  $F_{j-2}$  and  $F_{j-1}$  agree in  $U_{j-2}$ , and therefore we have

$$(45) \quad F_{j-1}(Q_w \cap U_{j-2}) \subset Q'_{w'}.$$

From (iii),  $F_{j-1}$  maps all components of  $Q_w \cap U_{j-1}$  into the same Segre variety. Therefore (45) implies  $F_{j-1}(Q_w \cap U_{j-1}) \subset Q'_{w'}$ , which by definition means that  $(w, w') \in A_j$ . In particular, the set  $A_j$  is non-empty. Condition (i) for  $A_j$  will be completely proved, once we know that  $A_j$  is a complex-analytic set,  $\dim A_j \equiv n$ , and select only the irreducible components of  $A_j$  which have intersection with  $A_{j-1}$  of dimension  $n$ .

Proof of (ii). Let  $(w^0, w'^0) \in A_j$ . If  $Q_w \cap U_{j-1}$  is connected for all  $w$  sufficiently close to  $w^0$ , then the proof of the fact that  $A_j$  is complex-analytic near  $(w^0, w'^0) \in A_j$  is the same as for  $A_1$  in Step 2. Let  $\tilde{U}$  be the largest connected relatively open subset of  $U_j$  such that  $0 \in \tilde{U}$  and for all  $z \in \tilde{U}$ ,  $F_{j-1}$  maps  $Q_z \cap U_{j-1}$  into the same Segre variety. From property (iii) for  $A_{j-1}$  we have  $U_{j-1} \subset \tilde{U}$ . Then (44) defines a holomorphic correspondence  $\tilde{A} \subset \tilde{U} \times \mathbb{P}^N$ . The proof is the same

as in Step 2 for  $A_1$ . Denote by  $\tilde{F}$  the multiple valued map associated with  $\tilde{A}$ . By repeating the argument used for  $A_1$ , we can show that for any  $w \in \tilde{U}$ ,  $\tilde{F}$  maps all connected components of  $Q_w \cap \tilde{U}$  into the same Segre variety.

For each point  $p \in \tilde{U}$  we may define now the following set

$$(46) \quad A_p = \{(w, w') \in U(Q_p) \times \mathbb{P}^N : \tilde{F}(Q_w \cap U_p) \subset Q'_{w'}\},$$

where the neighborhoods  $U(Q_p)$  of  $Q_p$  and  $U_p$  of  $p$  are chosen as in the construction of the set defined by (42). Let  $F_p$  be the map associated with  $A_p$ . Clearly,  $F_p$  coincides with  $F_{j-1}$  for  $p$  sufficiently close to the origin. We claim that the map  $F_p$  agrees with  $F_{j-1}$  in  $U(Q_p) \cap U_{j-1}$  for all  $p \in \tilde{U}$ . Indeed, suppose that  $w \in U(Q_p) \cap U_{j-1}$  and  $(w, w') \in A_{j-1}$ . Then  $F_{j-2}(Q_w \cap U_{j-2}) \subset Q'_{w'}$ . To prove the assertion we need to show that

$$(47) \quad \tilde{F}(Q_w \cap U_p) \subset Q'_{w'}.$$

Let  $z \in Q_w \cap U_p$  be an arbitrary point, and  $z' \in \tilde{F}(z)$ . Then  $F_{j-1}(Q_z \cap U_{j-1}) \subset Q'_{z'}$ . Since  $\tilde{F}$  and  $F_{j-1}$  agree in  $U_{j-1}$ , it follows that  $\tilde{F}(Q_z \cap U_{j-1}) \subset Q'_{z'}$ . For  $z \in \tilde{U}$ ,  $\tilde{F}$  maps different components of  $Q_z \cap \tilde{U}$  into the same Segre varieties, and therefore we have  $\tilde{F}(Q_z \cap \tilde{U}) \subset Q'_{z'}$ . In particular,  $w' \in Q'_{z'}$ , which implies  $z' \in Q'_{w'}$ . Since  $z$  was arbitrary, (47) holds.

By the construction,  $F_p$  maps  $Q_p$  into the same Segre variety for all  $p \in \tilde{U}$ . By analyticity this means that for any point  $w$  in  $\partial\tilde{U} \cap U_j$ ,  $F_{j-1}$  maps  $Q_w \cap U_{j-1}$  into the same Segre variety. Therefore,  $\tilde{U} = U_j$ . We choose only irreducible components of  $A_j$  of dimension  $n$  which contain  $A_{j-1}$ . This proves (ii) and also completes the proof of (i).

Finally, property (iii) can be shown the same way as it was done for  $A_1$ .

By the construction, from minimality of  $M$  and from [BER1], for some  $j_0 > 1$ , the set  $A_{j_0}$  defines a holomorphic correspondence  $F_{j_0}$  in a neighborhood  $\Omega_0 \subset U_{j_0}$  of the origin. Note that the size of this neighborhood depends only on the geometry of  $M$  and is independent of  $U_0$ , where  $f$  was originally defined. It remains to show now that  $F_{j_0}$  satisfies

$$(48) \quad F_{j_0}(M \cap \Omega_0) \subset M'.$$

If  $(z, z') \in A_{j_0}$ , then  $F_{j_0-1}(Q_z \cap U_{j_0-1}) \subset Q'_{z'}$ . From property (i),  $F_{j_0}(Q_z \cap U_{j_0-1}) \subset Q'_{z'}$ , and from (iii) it follows that

$$(49) \quad F_{j_0}(Q_z \cap U_{j_0}) \subset Q'_{z'}.$$

Suppose now that for some  $z^0 \in M \cap \Omega_0$ ,  $F_{j_0}(z^0) \not\subset M'$ . Then there exists  $z' \in F_{j_0}(z^0) \setminus M'$ . Note that  $F_{j_0}(M \cap U_0) \subset M'$ , and therefore by continuity we may find  $z^0$  and  $z'$  such that  $z'$  is close to  $M'$ . From (49) we have  $F_{j_0}(Q_{z^0} \cap U_{j_0}) \subset Q'_{z'}$ . Since  $z^0 \in Q_{z^0}$ , we have  $F_{j_0}(z^0) \subset Q'_{z'}$ , in particular,  $z' \in Q'_{z'}$ . But from (15),  $z' \notin Q'_{z'}$ , since  $z' \notin M'$ . This contradiction proves (48).

Theorem 2.1 is proved. Note that in general,  $F_{j_0}$  may not be finite-valued. However, by the Cartan-Remmert theorem (see e.g. [Lo]) combined with Remmert's proper mapping theorem, the set of points

$$(50) \quad \Sigma = \{z \in U_{j_0} : \dim \pi^{-1}(z) > 0\}$$

is a complex subvariety of  $U_{j_0}$ . Since  $\dim A_{j_0} \equiv n$ , we have  $\dim \Sigma < n$ , in particular,  $\Sigma$  does not divide  $U_{j_0}$ , and  $A_{j_0}|_{(U_{j_0} \setminus \Sigma) \times \mathbb{P}^N}$  is a finite-valued holomorphic correspondence.

5. EXTENSION AS A FINITE CORRESPONDENCE ALONG  $M$ .

In this section we give the proof of Theorem 2.2. One difficulty in the proof of analytic continuation of correspondences lies in the fact that the continued correspondence may acquire additional branches. To deal with this we define the notion of a *complete* correspondence as follows.

**Definition 5.1.** *Let  $M \subset \mathbb{C}^n$  be a smooth real-analytic essentially finite generic CR submanifold, and let  $M' \subset \mathbb{P}^N$  be a smooth compact real-algebraic essentially finite generic submanifold. Let  $F : U \rightarrow \mathbb{P}^N$  be a holomorphic correspondence such that  $F(U \cap M) \subset M'$ . Then  $F$  is called complete if for every  $z \in M$ , we have  $F(z) = \tilde{I}'_{z'}$  for some (and therefore for any)  $z' \in F(z)$ . Here  $\tilde{I}'_{z'}$  is defined as in (29).*

Note that since  $M'$  is essentially finite, a complete correspondence is finite-valued near  $M$ , but in general it may be reducible, even if defined on all of  $M$ .

Assume that  $M$  is locally, near  $p \in M$ , generically embedded into an open set in  $\mathbb{C}^n$ , so  $n = m + d$ . Let  $f$  be a holomorphic map defined in a neighborhood  $U_p \subset \mathbb{C}^n$  of  $p \in M$ , of maximal rank and such that  $f(U_p \cap M) \subset M'$ . We replace  $f$  with a complete correspondence. For that we choose a small neighborhood  $U_0$  of  $p$  and shrink  $U_p$  in such a way, that for any  $w \in U_0$ , the set  $Q_w \cap U_p$  is non-empty and connected. We define  $A_0$  as in (34). Denote the corresponding multiple valued map by  $F_0$ . Then  $F_0$  is a complete holomorphic correspondence. Indeed, since  $f$  is of maximal rank and  $d' \geq d$ , for  $z \in U_0$ ,  $\dim f(Q_z \cap U_p) = \dim Q_z = m$ . For  $z$  sufficiently close to  $p$ ,  $f(z)$  is close to  $M'$ , and since  $M'$  is essentially finite, the only points whose Segre varieties can contain  $f(Q_z \cap U_p)$  are in  $I'_{f(z)}$ . Furthermore,  $F_0$  is a finite correspondence. To see this we observe that if  $E \subset U_0$  is a positive dimensional set such that  $F_0(E)$  is discrete, then by the construction  $f(Q_z \cap U_p) \subset Q'_{z'}$  for all  $z \in E$ . But since  $\bigcup_{z \in E} Q_z \cap U_p$  has dimension bigger than  $m$ , this contradicts the fact that  $f$  is of maximal rank in  $U_p$ . Therefore the pre-image of any point under  $F_0$  is finite, and thus  $F_0$  is a finite correspondence.

We now show that  $F_0$  extends as a finite correspondence along any path on  $M_1$ . Our construction of the analytic continuation of  $F_0$  will preserve completeness, and therefore from Lemmas 3.1 and 3.3 we conclude that such analytic continuation will have the same number of branches near  $M_1$ . The problem can be localized as follows. Let  $\gamma : [0, 1] \rightarrow M_1$  be the given path,  $\gamma(0) = p$ , and assume that for  $t_0 \leq 1$ ,  $q = \gamma(t_0)$  is the first point on  $\gamma$  to which  $F_0$  does not extend as a finite correspondence. We choose  $t_1 \in [0, t_0]$  so close to  $t_0$ , that for  $a = \gamma(t_1)$  we have  $q \in \Omega_a$ , where  $\Omega_a$  is defined as in (21). Then by Theorem 2.1,  $F_0$  extends as a holomorphic correspondence  $A \subset \Omega_a \times \mathbb{P}^N$ . Thus we only need to show that after possibly shrinking the neighborhood  $\Omega_a$ , the set  $A$  is a finite correspondence. Let  $\pi : A \rightarrow \Omega_a$  and  $\pi' : A \rightarrow \mathbb{P}^N$  be the natural projections, and set  $F := \pi' \circ \pi^{-1}$ .

From (48) there exists a neighborhood  $\tilde{U} \subset \Omega_a$  of  $M \cap \Omega_a$  such that  $F(\tilde{U}) \subset U'$ , where  $U' \subset \mathbb{P}^N$  is a neighborhood of  $M'$  as in Lemma 3.1. We now repeat the argument of analytic continuation of  $A_0$  along  $\gamma$  by constructing the sets  $A_j^*$ ,  $j = 1, 2, \dots$ , with the only difference that the standard neighborhoods of the form (17) are chosen so small that they are contained in  $\tilde{U}$ . Since the new set  $\tilde{\Omega}_a$  may be smaller than  $\tilde{U}$ , this continuation may require more than one step. More precisely, we choose a sequence of points  $\{a_\nu\}_{\nu=0}^l$  such that  $a_\nu \in \gamma$ ,  $a_0 = a$ ,  $a_l = q$  and  $a_\nu \in \tilde{\Omega}_{a_{\nu-1}}$  for  $0 < \nu \leq l$ . For each  $a_\nu$  starting with  $a_0$  we use Theorem 2.1 to extend a finite correspondence  $F_\nu$  defined in a neighborhood of  $a_\nu$  to a holomorphic correspondence  $F_{\nu+1}$  defined in  $\Omega_{a_\nu} \subset \tilde{U}$ . This time we show in addition that at each step the extension is a finite correspondence. Since  $a_{\nu+1} \in \tilde{\Omega}_{a_\nu}$ , the process can be continued until we reach the point  $q$ .

Suppose that  $F_{\nu+1} : \Omega_{a_\nu} \rightarrow \mathbb{P}^N$  is a holomorphic correspondence, which is obtained from the finite correspondence  $F_\nu$  defined in a small neighborhood of the point  $a_\nu$  for some  $\nu$  by the inductive construction of the sets  $A_j^*$ ; that is

$$(51) \quad A_j^* = \{(w, w') \in U_j^* \times \mathbb{P}^N : F_{j-1}^*(Q_w \cap U_{j-1}^*) \subset Q_{w'}'\},$$

where the Segre set  $Q_{a_\nu}^j$  is contained in  $U_j^*$ , the map  $F_\nu$  is associated with the set  $A_0^*$ , and  $F_{\nu+1}$  corresponds to the set  $A_{j_0}^*$  with  $U_{j_0}^* = \Omega_{a_\nu}$ . Let  $F_j^*$  be the map associated with  $A_j^*$ .

Clearly, the sets  $A_j^*$  are contained in the set  $A$ , which defines a correspondence  $F : \tilde{U} \rightarrow U'$ . We claim that  $F_{\nu+1}$  is a finite correspondence. To prove this assertion we let  $k$  be the smallest integer such that  $F_k^*$  is not a finite correspondence. By assumption,  $k > 0$ . Suppose that there exists a point  $w' \in U'$  such that the analytic set  $F_k^{*-1}(w') \subset U_k^*$  has positive dimension. By the construction we have

$$(52) \quad F_{k-1}^*(Q_z \cap U_{k-1}^*) \subset Q_{w'}', \quad \text{for all } z \in F_k^{*-1}(w').$$

Since  $M$  is essentially finite, there exists a set  $E \subset F_k^{*-1}(w')$  such that  $\dim \cup_{z \in E} Q_z = m + 1$ . It follows from (52) that  $\dim F_{k-1}^*(\cup_{z \in E} Q_z) \leq m$ . But this contradicts the induction hypothesis that  $F_{k-1}^*$  is finite.

Suppose now that there exists a point  $w \in U_k^*$  such that  $F_k^*(w)$  is not discrete. By the construction this means that  $F_{k-1}^*(Q_w \cap U_{k-1}^*) \subset Q_{z'}'$ , where  $z'$  belongs to the non-discrete set. Since  $F_{k-1}^*$  is finite,  $F_{k-1}^*(Q_w \cap U_{k-1}^*)$  has dimension  $m$ . But then there can be only finitely many  $z'$  whose Segre varieties can contain  $F_{k-1}^*(Q_w \cap U_{k-1}^*)$ . Therefore  $F_k^*$  is also finite for any  $k \geq 0$ .

Hence, the obtained extension  $F_{\nu+1}$  is a finite correspondence. We can repeat the same argument for extending  $F_0$  along  $\gamma \cap \tilde{U}$  until we reach the point  $q$ .

Thus we have proved that  $F_0$  extends along any path on  $M_1$  as a finite correspondence  $F$ . It follows from the construction that the number of branches of  $F$  coincides with the number  $R$  defined in Lemma 3.1 for almost all points on  $M_1$ . It remains to observe now that from the simple connectivity of  $M_1$  and the monodromy theorem, the extension of  $F_0$  along homotopically equivalent paths gives the same result. The version of the monodromy theorem for finite-valued correspondences can be found in [Sh2].

Part (2) of the statement of the theorem follows from the construction of the extension  $F$ . Indeed, from the construction, if  $z \in M$ , and  $z' \in F(z)$ , then  $F(z) = \tilde{I}_{z'}$ . If  $\mathcal{X}$  is locally injective near  $z'$ , then  $F$  splits into  $R$  holomorphic mappings near  $z$ .

The proof of Theorem 2.2 is now complete.

## 6. PSEUDOCONCAVE SUBMANIFOLDS IN $\mathbb{P}^n$ .

In this section we prove the rest of the results stated in Section 2.

*Proof of Theorem 2.3.*

- (a) According to Thm 5.2 of [HN5], there exists an  $m + d$  dimensional irreducible algebraic subvariety  $Y$  of  $\mathbb{P}^n$  such that  $M$  is a generic CR submanifold of the regular part of  $Y$ ,  $\text{reg } Y$ . Because of the pseudoconcavity of  $M$  (or because of Property E), the continuous CR map  $f$  is smooth and has a unique holomorphic extension to an open neighborhood  $\mathcal{O}$  of  $M$  in  $\text{reg } Y$ . Thus  $f$  can be regarded as a holomorphic map from  $\Omega$  to  $\mathbb{P}^N$ . This means that  $f$  may be given by  $N$  meromorphic functions  $f_1, f_2, \dots, f_N$  in  $\Omega$ . To see this, we choose homogeneous coordinates  $[z_0 : z_1 : \dots : z_N]$  in  $\mathbb{P}^N$  such that the hyperplane  $\{z_0 = 0\}$  is in general position with respect to  $f(\mathcal{O})$ , and set  $\mathcal{O}_j = f^{-1}(V_j)$ , where  $V_j = \mathbb{P}^N \setminus \{z_j = 0\}$ ,  $j = 0, 1, \dots, N$ . The  $\mathcal{O}_j$  give an open cover of  $\mathcal{O}$ , in each  $V_j$  we have the inhomogeneous

coordinates  $(w_{1j}, w_{2j}, \dots, w_{Nj})$ , where  $w_{ij} = z_i/z_j$ , and  $f|_{\mathcal{O}_j}$  is given by holomorphic functions

$$(53) \quad \Omega_j \ni t \rightarrow (w_{1j}(t), w_{2j}(t), \dots, w_{Nj}(t)).$$

We define the meromorphic functions  $f_1, f_2, \dots, f_N$  by  $f_k(t) = w_{k0}(t)$  in  $\Omega_0$ , and by  $f_k(t) = w_{kj}(t)/w_{0j}(t)$  in  $\Omega_j$  for  $j, k = 1, 2, \dots, N$ . Note that these definitions are consistent on the overlaps.

By Thm 5.2 of [HN5], each  $f_k$  is the restriction to  $M$  of a rational function on  $Y$ , and hence can be regarded as a rational function on  $\mathbb{P}^n$ . This gives the desired rational map from  $\mathbb{P}^n$  to  $\mathbb{P}^N$ .

- (b) Since  $M$  is generic in  $\mathbb{P}^n$ , and since  $f$  is a local CR diffeomorphism, the Jacobian  $\det J_f$  of the extension of  $f$  to  $\mathbb{P}^n$  is not identically zero. Hence, the set  $\Sigma = \{z \in \mathbb{P}^n : \det J_f(z) = 0\}$  if non-empty, is a subvariety of  $\mathbb{P}^n \setminus L$  of complex codimension one, where  $L$  is the indeterminacy locus of  $f$ . Suppose  $\Sigma \neq \emptyset$ . Then, since  $f$  is locally biholomorphic near any point on  $M$ ,  $M \cap \Sigma = \emptyset$ . On the other hand, by the Remmert-Stein theorem, the closure of  $\Sigma$  is a subvariety of  $\mathbb{P}^n$  of codimension one. It is well known that its complement in  $\mathbb{P}^n$  is therefore a Stein manifold. But a pseudoconcave  $M$  (or an  $M$  satisfying Property E) has no CR embedding into a Stein manifold (see [HN1] and [HN5]). Thus  $\Sigma = \emptyset$ .

Let  $F : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$  be a polynomial map such that  $f \circ \pi = \pi \circ F$ , where  $\pi : \mathbb{C}^{n+1} \rightarrow \mathbb{P}^n$  is the canonical projection. Without loss of generality assume that the components of  $F$  are homogeneous polynomials of degree  $k$  without common factors. We claim that  $\det J_F(z) \neq 0$  for any point  $z \in \mathbb{C}^{n+1}$ . Indeed, suppose on the contrary that

$$(54) \quad E = \{z \in \mathbb{C}^{n+1} : \det J_F(z) = 0\}$$

is a non-empty subvariety of complex codimension one. Then  $F(E) \neq \{0\}$ , and therefore there exists a point  $p \in E$  such that  $F(p) \neq 0$ . For  $z \in \mathbb{C}^{n+1}$ , let  $L_z := \{lz : l \in \mathbb{C}\}$  be the complex line passing through the point  $z$  and the origin. Since the Jacobian of  $f$  does not vanish outside the indeterminacy locus, and  $F(L_p) \neq \{0\}$ , there exists a small neighborhood  $U$  of  $p$  such that for all  $z$  and  $w$  in  $U$ ,  $z \neq w$ ,

$$(55) \quad F(L_z) \cap F(L_w) = \{0\}.$$

Furthermore,  $F|_{U \cap L_z} = l^k z$ , and after shrinking  $U$  if necessary, we may assume that  $F|_{U \cap L_z}$  is an injective function for all  $z \in U$ . From this and (55) we conclude that  $F$  is injective in  $U$ , which contradicts the assumption that  $p \in E$ . Thus  $\det J_F \neq 0$  and therefore is a constant.

Finally, observe that  $\det J_F(z)$  is a homogeneous polynomial of degree  $(k-1)(n+1)$ , and being constant means that  $k = 1$ , i.e.  $F$  is a linear automorphism. □

*Proof of Theorem 2.4 and Corollary 2.5.* We may regard  $M$  as being a generic CR submanifold of a complex manifold  $X$ . Note that the pseudoconcavity of  $M$  (or Property E) implies that  $M$  is minimal, because minimality is well-known to be equivalent to wedge extendability. There is a neighborhood  $V_p$  of  $p$  in  $X$  such that  $f$  extends to a holomorphic mapping  $f : V_p \rightarrow \mathbb{P}^N$ . It is easy to check that, by possibly shrinking  $V_p$ , the extended map  $f$  has maximal rank in  $V_p$ . Hence we may conclude from Theorem 2.2 that  $f$  extends to a finite holomorphic correspondence  $F : V \rightarrow \mathbb{P}^N$ , where  $V \subset X$  is some neighborhood of  $M$ .

Since the Segre map associated with  $M'$  is injective,  $F$  splits at every point of  $M$ , and every map  $F^j$  of the splitting is a CR map from  $M$  to  $M'$ . One of them, say  $F^1$  is the extension of  $f$ .



By Theorem 2.3(a),  $F^1$  extends to a rational map from  $\mathbb{P}^n$  to  $\mathbb{P}^N$ . Moreover, if  $n = N$  and  $M$  is generic in  $\mathbb{P}^n$ , then  $F^1$  is locally biholomorphic, because the Segre map associated with  $M$  is injective. Thus by Theorem 2.3(b)  $F^1$  extends to a linear automorphism of  $\mathbb{P}^n$ . Rationality of  $F^1$  implies that  $M$  must also be algebraic, which proves Corollary 2.5.  $\square$

*Proof of Theorem 2.6.* Since  $M$  and  $M'$  are Levi non-degenerate, the associated Segre maps are locally injective, and  $M$  and  $M'$  satisfy the conditions of Theorem 2.2. If  $M$  and  $M'$  are strictly pseudoconvex then  $f$  extends as a locally biholomorphic map to a neighborhood of  $p$  by [PT]. If  $M$  is pseudoconcave, the result follows from [HN2]. Therefore the map  $f$  defined near  $p$  also satisfies the conditions of Theorem 2.2. Thus  $f$  extends as a finite correspondence along  $M$ . Since the set  $\Sigma'$ , where the Segre map associated with  $M'$  branches, is empty, the extended correspondence is single-valued. Since the Segre map associated with  $M$  is injective, the extension  $f : M \rightarrow M'$  is a locally biholomorphic map.

We now show that the extension is globally biholomorphic in a neighborhood of  $M$ . For that we note that  $M'$  is simply connected. Indeed, if  $k = n - 1$  or  $k = 0$ , then  $M' = S^{2n-1}$  which is simply connected. If  $0 < k < n - 1$ , then we choose an affine patch  $V'_n$  in  $\mathbb{P}^n$  where  $z'_n \neq 0$ . Then

$$(56) \quad M' \cap V'_n = \{|w'_0|^2 + \cdots + |w'_k|^2 - |w'_{k+1}|^2 - \cdots - |w'_{n-1}|^2 = 1\},$$

where  $w'_j = z'_j/z'_n$ . Let  $\pi$  be the projection from  $M'$  to the coordinates  $(w'_{k+1}, \dots, w'_{n-1})$ . Then  $\pi$  is onto, and for any  $w = (w'_{k+1}, \dots, w'_{n-1})$ ,  $\pi^{-1}(w) \cong S^{2k+1}$ , which is simply connected. Therefore,  $M'$  is also simply connected. Because of that the germ of a map  $f^{-1}$  extends holomorphically along any path on  $M'$  to a holomorphic map  $f^{-1} : M' \rightarrow M$ . Thus  $f|_M$  maps  $M$  one-to-one and onto  $M'$ , and  $f$  is globally biholomorphic.

If  $M'$  is pseudoconcave, then by the invariance of the Levi form, so is  $M$ . Thus by Theorem 2.3(b)  $f$  extends to a linear automorphism of  $\mathbb{P}^n$ .  $\square$

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